

Uncertainty of a function of uncertain variables

BY RUDOLF SÝKORA

16. 12. 2024

The (absolute) uncertainty of a function f of statistically independent (!) variables x_i is *generally* given by¹

$$\sigma_f = \sqrt{\sum_i (\partial_i f \sigma_i)^2},$$

where σ_i is the (absolute) uncertainty of x_i and $\partial_i f \equiv \frac{\partial f}{\partial x_i}$. By uncertainties here we mean the square roots of variables' *variances* ($\equiv \sigma^2$). Such uncertainties are called *standard*.

All other 'laws' to calculate the uncertainty of f are just special cases of the above.

Example 1. $f(x, y) = x y$

$$\begin{aligned}\sigma_f &= \sqrt{(y \sigma_x)^2 + (x \sigma_y)^2} = \sqrt{(x y)^2 \left(\left(\frac{\sigma_x}{x} \right)^2 + \left(\frac{\sigma_y}{y} \right)^2 \right)} = \\ &= f \sqrt{\left(\frac{\sigma_x}{x} \right)^2 + \left(\frac{\sigma_y}{y} \right)^2},\end{aligned}$$

i.e.,

$$\frac{\sigma_f}{f} = \sqrt{\left(\frac{\sigma_x}{x} \right)^2 + \left(\frac{\sigma_y}{y} \right)^2},$$

saying that *in this case* the relative uncertainty of f is given by the square root of sum of squares of the relative uncertainties of x and y .

Example 2. $f = \frac{\pi}{4}(D^2 - d^2)$

$$\sigma_f = \sqrt{\left(\frac{\pi}{4} 2 D \sigma_D \right)^2 + \left(\frac{\pi}{4} 2 d \sigma_d \right)^2} = \frac{\pi}{4} \sqrt{(2 D \sigma_D)^2 + (2 d \sigma_d)^2}.$$

Note that $2 D \sigma_D = D^2 \left(2 \frac{\sigma_D}{D} \right)$ is the absolute uncertainty of D^2 : $\frac{\sigma_D}{D}$ is the relative uncertainty of D and $2 \frac{\sigma_D}{D}$ is the relative uncertainty of $D^2 = D D$. Note especially that in the last case the relative uncertainty is $\frac{\sigma_D}{D} + \frac{\sigma_D}{D} = 2 \frac{\sigma_D}{D}$ and *not* $\sqrt{\left(\frac{\sigma_D}{D} \right)^2 + \left(\frac{\sigma_D}{D} \right)^2} = \sqrt{2} \frac{\sigma_D}{D}$, since the two terms are in this case 100% correlated and thus not statistically independent, quite the opposite. The general formula takes care of this automatically.

NB. Application of the general formula is (sort of) foolproof, you do not have to think much (relative/absolute?), but its application may often not be the simplest thing to do, nor it teaches you how things 'work'.

1. It is better to think about the formula in terms of variances, which add together to give the variance of the result (that is how one actually proves the formula):

$$\sigma^2[f(x, y, \dots)] = \left(\frac{\partial f}{\partial x} \right)^2 \sigma_x^2 + \left(\frac{\partial f}{\partial y} \right)^2 \sigma_y^2 + \dots$$